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ROYAL AIRCRAFT ESTABLISHMENT

TECHNICAL REPORT No. 64078

ON THE MOTION OF NEARLY SYNCHRONOUS SATELLITES

by

R. R. Allan

MINISTRY OF AVIATION
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SUMMARY

The longitudinal forces due to the longitude-dependent part of the Earth's gravitational potential can have considerable effects on nearly synchronous satellites. As a preliminary the special case of the circular equatorial orbit is considered. An approximate solution is given for the motion under the combination of all tesseral harmonics, and the relation to the Jacobi integral is discussed. For general orbits the disturbing function for the combination of all tesseral harmonics is developed in terms of the usual elliptic elements and the resonant terms identified and isolated. With a suitable definition of the mean longitude, it is shown that the motion in longitude relative to the Earth is equivalent to that of a particle moving in a one-dimensional potential, provided only that the eccentricity is small.

Departmental Reference: Space 81

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1 INTRODUCTION

Although various authors have discussed the motion of nearly synchronous satellites of the earth, only Musen and Bailie¹ and Morando^{2,3} have considered other than a circular equatorial orbit, the so-called 'geostationary' satellite. The treatment given by Musen and Bailie is valid for large eccentricities and all inclinations but includes only the $J_{2,2}$ tesseral harmonic. Morando, using von Zeipel's method as modified by Hori to deal with resonance, has given the stable positions and the periods of libration for each of the tesseral harmonics separately up to $\ell = 4$, $m = 4$. In a previous paper⁴ some results were given for the circular equatorial orbit which hold for the superposition of all tesseral harmonics, and the object of the present paper is to extend these results to more general orbits and in particular to nearly circular but inclined orbits.

2 THE GRAVITATIONAL POTENTIAL

We will write the gravitational potential $U(r, \theta, \phi)$ due to the earth at distance r from the centre of the earth, and at the point with colatitude θ and longitude ϕ , in the form

$$U(r, \theta, \phi) = U_0(r) + U'(r, \theta, \phi) \quad , \quad (1)$$

where $U_0 = \mu/r$ is the central potential, with $\mu = GM$ where M is the mass of the earth, and

$$U'(r, \theta, \phi) = (\mu/r) \sum_{\ell=2}^{\infty} \sum_{m=1}^{\ell} J_{\ell m} (R/r)^{\ell} P_{\ell}^m(\cos \theta) \cos m(\phi - \phi_{\ell m}) \quad . \quad (2)$$

Here R is the mean equatorial radius of the earth, and $J_{\ell m}$ and $\phi_{\ell m}$ are the constants associated with the (ℓ, m) tesseral harmonic. Also $P_{\ell}^m(z)$ is the associated Legendre function defined by

$$P_{\ell}^m(z) = (1-z^2)^{m/2} \frac{d^m}{dz^m} P_{\ell}(z) = (1-z^2)^{m/2} \frac{1}{2^{\ell} \ell!} \frac{d^{\ell+m}}{dz^{\ell+m}} (z^2-1)^{\ell} \quad . \quad \dots (3)$$

The terms with $m = 0$ would give the zonal harmonics, i.e. the axially symmetric part of the field, and are omitted. Also the possible term in (2) with $\ell = 2$, $m = 1$ must be very small since the axis of rotation of the earth must very nearly coincide with the principal axis⁵, and this term is also omitted.

In general the tesseral harmonics lead only to short-period variations; indeed, values of the constants $J_{\ell m}$ and $\varphi_{\ell m}$ have been determined from these short-period variations in the orbital elements of close-earth satellites. For a nearly synchronous satellite, however, the longitudinal forces due to the tesseral harmonics act continuously in the same sense and produce long-period changes in the energy, the semi-major axis, and the mean motion, thus leading to a libration in longitude. The main effect of the axially-symmetric terms which have been omitted from (1) and (2) (the only significant term is the second zonal harmonic which corresponds to the oblateness of the earth) is to produce a regression of the orbital plane which at synchronous height is of much longer period than the libration in longitude. The motion of distant circular orbits, including synchronous orbits, under the combined effect of the earth's oblateness and the luni-solar forces has recently been discussed by Allan and Cook⁶.

3 CIRCULAR EQUATORIAL ORBIT

If the orbit is circular and equatorial, the satellite will appear to be nearly stationary relative to the earth, apart from the long-period motion in longitude, and this special case is amenable to a simple treatment. We will ignore the small north-south component of force due to the odd tesseral harmonics (i.e. those with $\ell - m$ odd, which are antisymmetric about the equatorial plane) and assume that the satellite remains in the equatorial plane. Taking the polar coordinates of the satellite as r and φ , and the angular velocity of rotation of the earth as n_0 , the mean motion of the satellite is $n = n_0 + \dot{\varphi}$. Then the radial and transverse equations of motion can be written as

$$\ddot{r} - r(n_0 + \dot{\varphi})^2 = -\frac{\mu}{r^2} + F_r, \quad (4)$$

$$\frac{1}{r} \frac{d}{dt} \{r^2(n_0 + \dot{\varphi})\} = 2\dot{r}(n_0 + \dot{\varphi}) + r\ddot{\varphi} = F_\varphi, \quad (5)$$

where F_r and F_φ are the radial and transverse components of the disturbing force derived from (2). We may now expand about the equilibrium solution of these equations for zero perturbing force, namely

$$\dot{\phi} = 0, \quad r = a_0, \quad n_0^2 a_0^3 = \mu, \quad (6)$$

by writing $r = a_0 + \rho$, and retaining only terms of first order in ρ and $\dot{\phi}$. This procedure leads to the equations

$$\ddot{\rho} - 3n_0^2 \rho - 2a_0 n_0 \dot{\phi} = F_r, \quad (7)$$

$$2n_0 \dot{\rho} + a_0 \ddot{\phi} = F_\phi. \quad (8)$$

By eliminating ρ between (7) and (8) we can derive the following single equation for ϕ

$$a_0 \left\{ \ddot{\phi}^{(iv)} + n_0^2 \ddot{\phi}^{(ii)} \right\} = -3n_0^2 F_\phi - 2n_0 \frac{d}{dt} (F_r) + \frac{d^2}{dt^2} (F_\phi). \quad (9)$$

Since d/dt operating on F_r or F_ϕ is equivalent to $\dot{\phi}(\partial/\partial r) + \dot{\phi}(\partial/\partial \phi)$, the second term on the right of (9) is of higher order than the first term, and may be neglected. Similarly the third term on the right of (9) is smaller still, and is also neglected. Also provided the long-period motion in longitude is slow compared with the rotation of the earth, $\ddot{\phi}^{(iv)}$ on the left may also be neglected. Finally this leaves the approximate equation

$$a_0 \ddot{\phi} = -3F_\phi. \quad (10)$$

It is worth noting that according to (10), the apparent longitudinal acceleration relative to the earth in the long-period motion is three times as great as the true acceleration and in the opposite sense. Physically if the longitudinal force is in the direction of motion of the satellite ($F_\phi > 0$), the energy is increasing, so that the semi-major axis is also increasing and the mean motion is decreasing.

Equation (10) contains only the longitudinal component of the disturbing force in the equatorial plane, which is given by

$$F_\phi = \frac{1}{a_0} \frac{\partial}{\partial \phi} U(a_0, \pi/2, \phi), \quad (11)$$

where the disturbing potential is given as a general superposition of harmonics in (2). Then from (10) the equation of motion is

$$\ddot{\phi} = + 3n_0^2 \sum_m J_{\ell m} (R/a_0)^\ell P_\ell^m(0) \sin m(\varphi - \varphi_{\ell m}) \quad . \quad (12)$$

Since $P_\ell^m(0)$ vanishes when $(\ell - m)$ is odd, only the even tesseral harmonics contribute to the longitudinal disturbing force. Equation (12) may be integrated at once to give

$$\frac{1}{2} \dot{\phi}^2 + 3n_0^2 \sum J_{\ell m} (R/a_0)^\ell P_\ell^m(0) \cos m(\varphi - \varphi_{\ell m}) = \text{constant} \quad . \quad (13)$$

Thus the long-period motion in longitude relative to the rotating earth is equivalent to that of a particle moving with the velocity $a_0 \dot{\phi}$ in the one-dimensional gravitational potential $-3U'(a_0, \pi/2, \varphi)$.

Deduction from the Jacobi Integral - The result (13) can also be derived as an approximate form of the Jacobi integral of the system. Assuming the satellite remains in the equatorial plane so that we can set $\theta = \pi/2$ and $\dot{\theta} = 0$, the Lagrangian of the system may be written in terms of r and φ as

$$L = \frac{1}{2} \dot{r}^2 + \frac{1}{2} r^2 (n_0 + \dot{\phi})^2 + \mu/r + U'(r, \pi/2, \varphi) \quad . \quad (14)$$

Since the Lagrangian is time-independent, the system possesses the Jacobi integral, $H = \text{constant}$, which takes the form

$$H = \frac{1}{2} \dot{r}^2 + \frac{1}{2} r^2 (\dot{\phi}^2 - n_0^2) - \mu/r - U'(r, \pi/2, \varphi) = \text{constant} \quad . \quad (15)$$

We will now show that this reduces to the approximate result (13) to order J , where J stands for $J_{\ell m}$. First of all from (13) $\dot{\phi}$ is of order $J^{1/2}$, while from (12) $\ddot{\phi}$ is of order J , and in general the k th derivative of φ is of order $J^{k/2}$. On the other hand we can write equation (4) in the form

$$(n_0 + \dot{\phi})^2 r^3 = \mu + O(J) = n_0^2 a_0^3 + O(J) \quad , \quad (16)$$

which leads to

$$r = a_0 \left\{ 1 - \frac{2}{3} \frac{\dot{\phi}}{n_0} + \delta_1 + O(J^{3/2}) \right\} \quad , \quad (17)$$

where δ_1 is of order J . Consequently

$$\rho = -\frac{2}{3} a_0 \dot{\phi}/n_0 \{1 + O(J^{1/2})\} , \quad (18)$$

so that ρ is of order $J^{1/2}$, while $\dot{\phi}$ (which is also \dot{r}) is of order J and $\ddot{\phi}$ is of order $J^{3/2}$.

In (15) \dot{r}^2 may be neglected since it is of order J^2 , and on substituting from (17) for r^2 and r^{-1} the remainder gives

$$\begin{aligned} & -\frac{1}{2} n_0^2 a_0^2 (1 - \dot{\phi}^2/n_0^2) \left\{ 1 - \frac{4}{3} \frac{\dot{\phi}}{n_0} + \frac{4}{9} \frac{\dot{\phi}^2}{n_0^2} + 2\delta_1 + O(J^{3/2}) \right\} \\ & - n_0^2 a_0^2 \left\{ 1 + \frac{2}{3} \frac{\dot{\phi}}{n_0} - \delta_1 + \frac{4}{9} \frac{\dot{\phi}^2}{n_0^2} + O(J^{3/2}) \right\} - U'(a_0, \pi/2, \varphi) \\ & = \text{constant} + O(J^{3/2}) . \end{aligned}$$

After multiplying out the first product and adding, the terms involving δ_1 cancel, leaving the result

$$- n_0^2 a_0^2 \left\{ \frac{3}{2} + \frac{1}{6} \frac{\dot{\phi}^2}{n_0^2} + O(J^{3/2}) \right\} - U'(a_0, \pi/2, \varphi) = \text{constant} ,$$

i.e.

$$\frac{1}{2} a_0^2 \dot{\phi}^2 + 3U'(a_0, \pi/2, \varphi) = \text{constant} + O(J^{3/2}) , \quad (19)$$

which agrees with the previous result (13).

In the light of this consideration of orders of magnitude, it is clear that (10) is derived using only that part of (7) which is of order $J^{1/2}$, namely

$$- 3n_0^2 \rho - 2a_0 n_0 \dot{\phi} = 0 + O(J) . \quad (7')$$

In fact (7) is not accurate to order J , since among the quadratic terms neglected the following three are of order J :

$$- a_0 \dot{\phi}^2 , \quad - 2n_0 \rho \dot{\phi} , \quad + 3n_0^2 \rho^2/a_0 .$$

This, however, does not affect the results (10), (12) and (13).

4. DEVELOPMENT OF THE DISTURBING FUNCTION FOR GENERAL ORBITS

For a more general orbit, the motion in longitude is still controlled by the longitudinal force experienced by the satellite, and it can be shown⁴ that the motion must have the same general character of an oscillation in longitude even for orbits with considerable inclination and eccentricity. However it is not easy to evaluate the mean longitudinal force when the satellite is no longer nearly stationary relative to the earth.

It seems simplest to return to the disturbing potential (2) and express it in terms of the elements of the orbit. The shortest, and also the most general, development of the disturbing function seems to be that given by Izsak⁷. He points out that on rotating the coordinate system the ℓ th degree spherical harmonic $P_{\ell}^m(\cos \theta) \exp(im\phi)$ must transform to a linear combination of the ℓ th degree spherical harmonics $P_{\ell}^{m'}(\cos \theta') \exp(im'\phi')$ where θ' and ϕ' are the polar coordinates in the new system. If the new coordinate system is chosen with its pole OZ' coincident with the pole of the satellite orbit then $\theta' = \pi/2$ and the associated Legendre polynomials $P_{\ell}^{m'}(\cos \theta')$ reduce to constants. Moreover since $P_{\ell}^{m'}(0)$ vanishes when $\ell + m'$ is odd, the sum contains only terms with $\ell + m'$ even. The actual form of the result can be derived from texts on the applications of group theory in quantum mechanics (e.g. Wigner⁸ or Rose⁹), and can be written as

$$P_{\ell}^m(\cos \theta) \exp(im\phi) = \sum_{j=0}^{\ell} K_{\ell m}^{\ell-2j} \exp\{i[(\ell-2j)(\omega+f) + m(\Omega-n_0 t)]\}, \quad (20)$$

where

$$K_{\ell m}^{\ell-2j} = i^{\ell-m} \frac{(\ell+m)!}{2^{\ell} j!(\ell-j)!} \sum_k (-)^k \binom{2\ell-2j}{k} \binom{2j}{\ell-m-k} o^{3\ell-m-2j-2k} s^{-\ell+m+2j+2k} \dots \quad (21)$$

In (20), Ω is the right ascension of the ascending node, ω is the argument of perigee, f is the true anomaly, and $n_0 t$ is the current sidereal time of Greenwich, so that $\Omega - n_0 t$ is the angle from the current position of Greenwich to the ascending node and $\omega + f$ is the angle from the ascending node to the current position of the satellite (see Fig.1). Also in (21), $o = \cos I/2$ and $s = \sin I/2$ where I is the inclination, so that the K -coefficients depend only

on the inclination. The form $K_{\ell m}^{\ell-2j}$ quoted here is identical to Issak's $K_{\ell m}^j$ except that the factor $i^{\ell-m}$ has been included within the coefficient.

With the aid of (20) the disturbing potential (2) can be written as

$$U^* = \sum U_{\ell m}, \quad (22)$$

where the contribution of the (ℓ, m) th tesseral harmonic is

$$U_{\ell m} = (\mu/R) J_{\ell m} (R/r)^{\ell+1} \Re \sum_{j=0}^{\ell} K_{\ell m}^{\ell-2j} \exp\{i[(\ell-2j)(\omega+f) - m\varphi_{\ell m}^*]\} \quad \dots (23)$$

Here \Re denotes the real part. It has been convenient in (23) to introduce the quantity $\varphi_{\ell m}^*$ given by

$$\varphi_{\ell m}^* = \varphi_{\ell m} + n_0 t - \Omega, \quad (24)$$

which is the current longitude, relative to the ascending node, of the meridian $\varphi = \varphi_{\ell m}$ associated with the (ℓ, m) th tesseral harmonic.

The expression (23) still contains the true anomaly and must be developed in terms of the mean anomaly M according to

$$(r/a)^{-\ell-1} \exp(ipf) = \sum_{q=-\infty}^{\infty} X_q^{\ell-1,p} \exp(iqM), \quad (25)$$

which introduces the Hansen coefficients $X_q^{\alpha,p}$. These coefficients depend only on the eccentricity and can be expanded as power series in e , and we will recall the following properties:

$$(i) \quad \text{If } p = q, \quad X_q^{\alpha,p} = 1 + O(e^2).$$

$$(ii) \quad \text{If } p \neq q, \text{ the leading term in } X_q^{\alpha,p} \text{ is of order } |p-q| \text{ in } e.$$

Then from (23) and (25)

$$U_{\ell m} = (\mu/R) J_{\ell m} (R/a)^{\ell+1} \Re \sum_j \sum_q K_{\ell m}^{\ell-2j} X_q^{\ell-1, \ell-2j} \exp\{i[(\ell-2j)\omega + qM - m\varphi_{\ell m}^*]\}, \quad \dots (26)$$

which completes the development of the disturbing function in terms of the elements a, e, I, Ω, ω and χ' , where χ' is the modified mean anomaly at the epoch, defined by

$$M = \int_0^t n \, dt + \chi' \quad . \quad (27)$$

To pick out the terms which will be important for synchronous orbits, we note that the rate of change of the argument in (26) is

$$(\ell - 2j) \dot{\omega} + q(n + \dot{\chi}') - m(n_0 - \dot{\Omega}) = qn - m n_0 \quad . \quad (28)$$

Since $n = n_0$, the resonant terms are those with $q = m$, and the resonant part of the potential can be written as

$$U_{\ell m}^* = (\mu/R) J_{\ell m} (R/a)^{\ell+1} \sum_j K_{\ell m}^{\ell-2j} X_m^{\ell-1, \ell-2j} \exp\{i[(\ell-2j)\omega + m(M - \varphi_{\ell m}^*)]\} \quad . \quad \dots (29)$$

Since the Hansen coefficients in (29) are of order $|\ell - m - 2j|$ in e , the lowest order of term which actually occurs depends on whether $(\ell - m)$ is even or odd:

- (i) If $(\ell - m)$ is even, the lowest order term is given by $2j = \ell - m$ and is independent of e .
- (ii) If $(\ell - m)$ is odd, the lowest order terms are those of order e given by $2j = \ell - m \pm 1$.

5 MOTION UNDER THE SECOND SECTORIAL HARMONIC ALONE

It seems desirable to consider in more detail the term containing $J_{2,2}$ which corresponds to the ellipticity of the earth's equator. Although the coefficients $J_{\ell m}$ are not very well determined as yet it is probable that $J_{2,2}$ is the largest coefficient, and fairly certain that it will give the dominant term at synchronous height since $U_{\ell m}$ decreases with distance as $r^{-(\ell+1)}$. Thus from (29), evaluating the K -coefficients according to (21) and expanding the Hansen coefficients up to order e^4 , the resonant part of the disturbing function for this harmonic takes the form

$$\begin{aligned}
U_{2,2}^* &= 3(\mu/R) J_{2,2}(R/a)^3 \\
&\times \left\{ \left(1 - \frac{5}{2} e^2 + \frac{13}{16} e^4 + \dots \right) o^4 \cos 2(M - \varphi_{2,2}^* + \omega) \right. \\
&\quad + \left(\frac{9}{2} e^2 + \frac{7}{2} e^4 + \dots \right) o^2 s^2 \cos 2(M - \varphi_{2,2}^*) \\
&\quad \left. + \left(\frac{e^4}{24} + \dots \right) s^4 \cos 2(M - \varphi_{2,2}^* - \omega) \right\} . \quad (30)
\end{aligned}$$

This result could also be written down from the work of Cook¹⁰ who has given an explicit development of the disturbing function up to $\ell = m = 4$. He was, however, only concerned with the 'long-period' terms for non-resonant conditions and eventually discarded the terms which are important here.

The rates of change of all the elements, a, e, I, Ω, ω and χ^1 can now be derived from (30) by Lagrange's planetary equations¹¹. As will be shown, the motion of synchronous satellites is determined almost entirely by the variation in the semi-major axis, and the variations of all the other elements are unimportant. Consequently we give to order e^4 only the equation for da/dt . The Lagrange equation for a , namely

$$da/dt = (2/na)(\partial U^1/\partial \chi^1) , \quad (31)$$

leads to the result

$$\begin{aligned}
da/dt &= -12 na J_{2,2}(R/a)^2 \\
&\times \left\{ \left(1 - \frac{5}{2} e^2 + \frac{13}{16} e^4 + \dots \right) o^4 \sin 2(M - \varphi_{2,2}^* + \omega) \right. \\
&\quad + \left(\frac{9}{2} e^2 + \frac{7}{2} e^4 + \dots \right) o^2 s^2 \sin 2(M - \varphi_{2,2}^*) \\
&\quad \left. + \left(\frac{e^4}{24} + \dots \right) s^4 \sin 2(M - \varphi_{2,2}^* - \omega) \right\} . \quad (32)
\end{aligned}$$

Clearly the most useful variable to describe the motion is simply the combination

$$M - \varphi_{2,2}^* + \omega = (M + \omega + \Omega) - n_0 t - \varphi_{2,2}$$

which occurs in the argument of the resonant term of zero order in e in (30). The slowly-varying quantity $(M + \omega + \Omega) - n_0 t$ defines the mean longitude of the satellite relative to the earth, and we will write

$$\bar{\varphi} = (M + \omega + \Omega) - n_0 t = \int n dt - n_0 t + e' \quad (33)$$

where $e' = \chi' + \omega + \Omega$ is the modified mean longitude at the epoch.

It is fairly easy to interpret $\bar{\varphi}$ in terms of the positions of the nodes. For an exactly synchronous satellite ($n = n_0$) the ground-track on the earth is a fixed curve which degenerates to a symmetrical figure-of-eight centred on the equator if the orbit is also circular (see Fig.2). For an eccentric orbit the true anomaly is $-\omega$ at the ascending node ($t = t_N$), and $(\pi - \omega)$ at the descending node ($t = t_{N'}$), so that from Kepler's equation the mean anomaly takes the values $(-\omega + e \sin \omega)$ and $(\pi - \omega - e \sin \omega)$ at the ascending and descending nodes respectively. Using these values in (33) the longitudes of the ascending and descending nodes relative to the earth are respectively

$$\begin{aligned} \Omega - n_0 t_N &= \bar{\varphi} - e \sin \omega, \\ (\Omega + \pi) - n_0 t_{N'} &= \bar{\varphi} + e \sin \omega, \end{aligned}$$

i.e. $\bar{\varphi}$ is given by the mean of the ascending and descending nodes on the earth. In practice the satellite is not exactly synchronous so that $\bar{\varphi}$ is slowly-varying and the ground-track changes, but the value of $\bar{\varphi}$ at any instant can still be determined by interpolation

Using the mean longitude $\bar{\varphi}$ the rates of change of all the elements for a nearly circular orbit are as follows:

$$da/dt = -12 na J_{2,2} (R/a)^2 o^4 \sin 2(\bar{\varphi} - \varphi_{2,2}) + O(e^2)$$

$$de/dt = 3ne J_{2,2} (R/a)^2 \{ o^4 \sin 2(\bar{\varphi} - \varphi_{2,2}) - 9 o^2 s^2 \sin 2(\bar{\varphi} - \varphi_{2,2} - \omega) \} + O(e^3)$$

$$dI/dt = 6n J_{2,2} (R/a)^2 o^3 s \sin 2(\bar{\varphi} - \varphi_{2,2}) + O(e^2)$$

$$d\Omega/dt = -3n J_{2,2} (R/a)^2 o^2 \cos 2(\bar{\varphi} - \varphi_{2,2}) + O(e^2)$$

$$\begin{aligned} d\omega/dt = 3n J_{2,2}(R/a)^2 \{ & -o^2(1+3o^2) \cos 2(\bar{\varphi} - \varphi_{2,2}) \\ & + 9o^2 s^2 \cos 2(\bar{\varphi} - \varphi_{2,2} - \omega) \} + O(e^2) \end{aligned}$$

$$\begin{aligned} d\chi'/dt = 3n J_{2,2}(R/a)^2 \{ & 11o^4 \cos 2(\bar{\varphi} - \varphi_{2,2}) \\ & - 9o^2 s^2 \cos 2(\bar{\varphi} - \varphi_{2,2} - \omega) \} + O(e^2) \end{aligned} \quad (34)$$

There are terms in the last two equations of (34) which are of zero order in e yet still depend on ω , but these terms cancel on forming the equation for $\bar{\varphi}$. Thus from (33)

$$d\bar{\varphi}/dt = (n - n_0) + d\varepsilon'/dt,$$

and differentiating once more,

$$d^2\bar{\varphi}/dt^2 = dn/dt + d^2\varepsilon'/dt^2 \quad (35)$$

Then from the first of (34),

$$dn/dt = +18n^2 J_{2,2}(R/a)^2 o^4 \sin 2(\bar{\varphi} - \varphi_{2,2}) + O(e^2) \quad (36)$$

Likewise from the last three of (34),

$$d\varepsilon'/dt = 6n J_{2,2}(R/a)^2 o^2(4o^2 - 1) \cos 2(\bar{\varphi} - \varphi_{2,2}) + O(e^2), \quad (37)$$

and differentiating once more

$$\begin{aligned} d^2\varepsilon'/dt^2 = & -12n d\bar{\varphi}/dt J_{2,2}(R/a)^2 o^2(4o^2 - 1) \times \sin 2(\bar{\varphi} - \varphi_{2,2}) + O(e^2) \\ & \dots (38) \end{aligned}$$

Since $d\bar{\varphi}/dt \ll n$, the second term on the right of (35) is negligible compared with the first term. In fact (36) is of order J , while (38) is of order $J^{3/2}$ since $d\bar{\varphi}/dt$ is of order $J^{1/2}$. To lowest order the motion in mean longitude is controlled only by the variation in the semi-major axis, and if the second term in (35) is neglected the equation of motion can be written, to within order e^2 , as

$$d^2\bar{\varphi}/dt^2 = \frac{1}{2} k^2 \sin 2(\bar{\varphi} - \varphi_{2,2}) , \quad (39)$$

where

$$k^2 = 36n^2 J_{2,2} (R/a)^2 a^4 . \quad (40)$$

There are clearly four positions of equilibrium; the two positions $\bar{\varphi} = \varphi_{2,2} \pm \pi/2$ (on the minor axis of the earth's equatorial section) are stable, while those at $\bar{\varphi} = \varphi_{2,2}$ and $\bar{\varphi} = \varphi_{2,2} + \pi$ (on the major axis) are unstable. To integrate (39) it is more convenient to change the origin to one or other of the two stable positions by writing

$$\psi = \bar{\varphi} - \varphi_{2,2} \pm \pi/2 ,$$

so that (39) becomes

$$\ddot{\psi} = -\frac{1}{2} k^2 \sin 2\psi . \quad (41)$$

The first integral of (41) can be written as

$$\dot{\psi}^2 - k^2 \cos^2 \psi = \text{constant} = \dot{\psi}_0^2 - k^2 \cos^2 \psi_0 , \quad (42)$$

where ψ_0 and $\dot{\psi}_0$ are the initial values. The satellite will be captured in an effective potential well and will oscillate about one or other of the two points of stable equilibrium provided that

$$\dot{\psi}_0^2 < k^2 \cos^2 \psi_0 . \quad (43)$$

If this condition is satisfied, ψ will oscillate between the limits $\pm \psi_m$ say, where

$$k^2 \sin^2 \psi_m = \dot{\psi}_0^2 + k^2 \sin^2 \psi_0 , \quad (44)$$

and (42) can be written as

$$\dot{\psi}^2 = k^2 (\sin^2 \psi_m - \sin^2 \psi) . \quad (45)$$

From (45) the period of a complete oscillation is

$$T = \frac{4}{k} \int_0^{\psi_m} (\sin^2 \psi_m - \sin^2 \psi)^{-1/2} d\psi = \frac{4}{k} \int_0^{\pi/2} (1 - \sin^2 \psi_m \sin^2 u)^{-1/2} du .$$

... (46)

The result can be written in the alternative forms

$$T = \frac{4}{k} K(\sin \psi_m) = \frac{2\pi}{k} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \sin^2 \psi_m\right) , \quad (47)$$

where $K(\sin \psi_m)$ is a complete elliptic integral of the first kind, and ${}_2F_1$ is the ordinary hypergeometric function. If the amplitude of the oscillation is small, the period is approximately $2\pi/k$, which is about 780 days for an orbit of low inclination if we take $J_{2,2} \approx 2 \times 10^{-6}$. The period increases with the amplitude of the oscillation and approaches infinity as ψ_m approaches $\pi/2$.

To return to the variation of the remaining elements in (34), it is clear that de/dt is completely negligible. Both dI/dt and $d\Omega/dt$ must certainly be less than $3n J_{2,2} (R/a)^2$ which is approximately 5×10^{-5} degrees/day taking $J_{2,2} \approx 2 \times 10^{-6}$. The inclination oscillates as the satellite performs a libration around a position of stable equilibrium. The remaining quantities $d\omega/dt$ and dx'/dt are rather larger, and also more complicated; in particular there are parts of zero order in e which depend on the argument of perigee. However for the more meaningful combination ε' given in (37),

$$d\varepsilon'/dt \leq 18n J_{2,2} (R/a)^2 ,$$

which is approximately 3×10^{-4} degree/day, again taking $J_{2,2} \approx 2 \times 10^{-6}$. For the first term on the right of (35), we have from (36) that

$$dn/dt \leq 18n^2 J_{2,2} (R/a)^2 ,$$

which is approximately 2×10^{-3} degree/day². From (36) and (38) the ratio between the two terms on the right of (35), taking $I = 0$ for simplicity, is $(2/n) d\bar{\omega}/dt + O(e^2)$. Now from (45) $d\bar{\omega}/dt \leq k$, where $k \approx 10^{-7} \text{ sec}^{-1}$ for $J_{2,2} \approx 2 \times 10^{-6}$, so that the ratio is at most about 2.5×10^{-3} , and the second term on the right of (35) is only a small correction.

6 MOTION UNDER THE COMBINATION OF ALL TESSERAL HARMONICS FOR e SMALL

From the development of Section 4 and in particular from (29) et seq, the resonant part of the complete disturbing function to lowest (zero) order in e may be written as

$$U^* = (\mu/R) \sum' J_{\ell m}(R/a)^{\ell+1} K_{\ell m}^m \cos m(M + \omega - \varphi_{\ell m}^*) + O(e) \quad (48)$$

Here the dash denotes that the summation is only over the even tesseral harmonics; the part of order e arises from the odd tesseral harmonics. In writing (48), we have used the properties of the Hansen coefficients and also the fact that, according to (21), the K -coefficients are real for $\ell - m$ even. This disturbing function is then adequate to describe the motion for orbits at any inclination provided the eccentricity is small.

From (20) it is clear that when $I = 0$, $\theta = \pi/2$, and consequently $K_{\ell m}^{\ell-2j} = \delta_{\ell-2j,m} P_{\ell}^m(0)$ for $I = 0$. Thus it is possible to write, when $\ell - m$ is even,

$$K_{\ell m}^m(I) = D_{\ell m}(I) P_{\ell}^m(0) \quad (49)$$

where each $D_{\ell m} = 1$ when $I = 0$. Also from (24) and (33)

$$M + \omega - \varphi_{\ell m}^* = \bar{\varphi} - \varphi_{\ell m} \quad (50)$$

so that the resonant part of the disturbing function takes the form

$$U^* = (\mu/R) \sum' J_{\ell m}(R/a)^{\ell+1} P_{\ell}^m(0) D_{\ell m}(I) \cos m(\bar{\varphi} - \varphi_{\ell m}) + O(e) \quad (51)$$

where the sum is over the even tesseral harmonics. The coefficients $D_{\ell m}$ for the five even tesseral harmonics up to $\ell = m = 4$ are easily found from (21) or from the explicit development of Cook¹⁰ and are given in Table 1 together with the values of $P_{\ell}^m(0)$.

TABLE 1

The coefficients $D_{\ell m}$ up to $\ell = m = 4$

Harmonics	$P_{\ell}^m(0)$	$D_{\ell m}(I)$
(2,2)	3	$D_{2,2} = o^4$
(3,1)	-3/2	$D_{3,1} = o^2 (1 - 10 s^2 + 15 s^4)$
(3,3)	15	$D_{3,3} = o^6$
(4,2)	-15/2	$D_{4,2} = o^4 (1 - 14 s^2 + 28 s^4)$
(4,4)	105	$D_{4,4} = o^8$

As before the equation of motion for $\bar{\varphi}$ is given by (35) where the first and second terms on the right are of order J and $J^{3/2}$ respectively. Neglecting the higher order term and combining (35) and (31),

$$d^2 \bar{\varphi} / dt^2 = dn/dt = - (3/a^2) \partial U^* / \partial \bar{\varphi} \quad (52)$$

where $\partial / \partial \chi'$ in (31) has been replaced by $\partial / \partial \bar{\varphi}$ on account of (50). Then from (51) and (52),

$$d^2 \bar{\varphi} / dt^2 = 3n_0^2 \sum_{\ell m} J_{\ell m} (R/a)^{\ell} P_{\ell}^m(0) D_{\ell m}(I) \sin m(\bar{\varphi} - \varphi_{\ell m}) + O(e) \quad \dots (53)$$

In finding the equation of motion for the mean longitude, we have tacitly assumed that the odd tesseral harmonics, which are lumped together as the terms of order e in the potential, only contribute terms of order e to (53). This is fairly clear physically since the odd harmonics are antisymmetric about the equator. However we ought to consider the effect of the odd harmonics on all orbital elements as we have done in (34) for the $J_{2,2}$ term. Going back to (29) and using Lagrange's planetary equations, it is easily shown that the odd harmonics contribute terms of order e to da/dt , dI/dt and $d\Omega/dt$. As is to be expected the odd harmonics are more effective in changing the position of perigee and give terms of zero order in e in de/dt and of order e^{-1} in dn/dt and

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As before the equation of motion for $\bar{\varphi}$ is given by (35) where the first and second terms on the right are of order J and $J^{3/2}$ respectively. Neglecting the higher order term and combining (35) and (31),

$$d^2\bar{\varphi}/dt^2 = d\eta/dt = - (3/a^2) \partial U^*/\partial \bar{\varphi} \quad (52)$$

where $\partial/\partial \chi'$ in (31) has been replaced by $\partial/\partial \bar{\varphi}$ on account of (50). Then from (51) and (52),

$$d^2\bar{\varphi}/dt^2 = 3n_0^2 \sum_{\ell} J_{\ell m} (R/a)^{\ell} P_{\ell}^m(0) D_{\ell m}(I) \sin m(\bar{\varphi} - \varphi_{\ell m}) + O(e) \quad \dots (53)$$

In finding the equation of motion for the mean longitude, we have tacitly assumed that the odd tesseral harmonics, which are lumped together as the terms of order e in the potential, only contribute terms of order e to (53). This is fairly clear physically since the odd harmonics are antisymmetric about the equator. However we ought to consider the effect of the odd harmonics on all orbital elements as we have done in (34) for the $J_{2,2}$ term. Going back to (29) and using Lagrange's planetary equations, it is easily shown that the odd harmonics contribute terms of order e to da/dt , dI/dt and $d\Omega/dt$. As is to be expected the odd harmonics are more effective in changing the position of perigee and give terms of zero order in e in de/dt and of order e^{-1} in $d\omega/dt$ and

dx'/dt separately. However the contribution to the more meaningful combination dz'/dt is easily shown to be of order e . Thus the odd tesseral harmonics contribute only terms of order e to (53). On the other hand all the even tesseral harmonics give terms in da/dt , etc of the same form as in (34).

The result given in (53) is similar in form to (12) except that the longitude φ in (12) has been replaced by the more general quantity $\bar{\varphi}$, and (53) also contains the factors $D_{\ell m}(I)$. As for the geostationary case the motion in $\bar{\varphi}$ is equivalent to that of a particle moving in a one-dimensional potential and in principle the satellite could be captured in any potential minimum. Of course from the estimates which have so far been obtained for the coefficients $J_{\ell m}$, it is very probable that the potential is dominated by the (2,2) harmonic which gives just two minima in the potential. However it is just possible that the superposition of the other tesseral harmonics could give rise to further minima, particularly by splitting the principal minima of the (2,2) harmonic into a number of separate minima.

If the coefficients $D_{\ell m}(I)$ are assumed to be sufficiently constant over the period in question, (53) could be integrated to give

$$\frac{1}{2}(\frac{d\bar{\varphi}}{dt})^2 + 3n_0^2 \sum_{\ell} J_{\ell m} (R/a_0)^{\ell} P_{\ell}^m(0) D_{\ell m}(I) \cos m(\bar{\varphi} - \varphi_{\ell m}) = \text{constant} + O(e) \quad \dots (54)$$

However this integral is more suspect and less useful than the corresponding result (13) for a nearly equatorial satellite, since the inclination of the orbit is changing with time⁶. It is true that the inclination of an initially equatorial orbit also changes with time, but $\partial D_{\ell m}/\partial I$ vanishes for an equatorial orbit although not for an inclined orbit. In fact two of the coefficients in Table 1 change considerably with inclination; $D_{3,1}$ becomes negative when the inclination exceeds 40° and becomes positive again above 95° . Likewise $D_{4,2}$ has zeroes at 34° and 80° inclination. The remaining coefficients in Table 1 are positive for all inclinations.

Although the potential is very probably dominated by the (2,2) harmonic, it is still interesting to consider the motion under the general (ℓ, m) tesseral harmonic alone. From (53), if only the (ℓ, m) harmonic is present, there will be $2m$ positions of equilibrium divided into two sets each of m as follows:

$$\begin{aligned} (i) \quad \bar{\varphi} &= \varphi_{\ell m} + 2p \pi/m, \\ (ii) \quad \bar{\varphi} &= \varphi_{\ell m} + (2p+1) \pi/m, \end{aligned}$$

where $p = 0, 1, 2, \dots, (m-1)$. If the coefficient $J_{\ell m} P_{\ell}^m(0) D_{\ell m}(I)$ is positive, the former are unstable and the latter are stable, and vice-versa if the coefficient is negative. In any case the coefficient can be made positive by suitably redefining $\varphi_{\ell m}$ so as to change the sign of $J_{\ell m}$. Assuming the coefficient is positive, the substitution

$$\bar{\varphi} - \varphi_{\ell m} = 2\psi/m + (2p+1)\pi/m, \quad (55)$$

changes the origin to a stable point, and reduces the equation of motion to

$$\ddot{\psi} = -\frac{1}{2} k_{\ell m}^2 \sin 2\psi + O(e), \quad (56)$$

where

$$k_{\ell m}^2 = 3n_0^2 m^2 J_{\ell m} (R/a_0)^{\ell} P_{\ell}^m(0) D_{\ell m}(I). \quad (57)$$

Since (56) is equivalent to (41), the equations (42) to (47) hold with k replaced by $k_{\ell m}$.

7 CONCLUSIONS

As is now well-known the very small longitudinal forces due to the slight asymmetry of the Earth about its axis can have considerable effects on nearly synchronous satellites. Because these forces can act continuously in one sense, the energy of the satellite in its motion around the Earth slowly changes. As a preliminary some further consideration has been given in the first part of this Report to the special case of a satellite in a circular equatorial orbit. Here, as has been shown before, the motion of the satellite relative to the Earth is simply that of a particle moving in a one-dimensional potential which, apart from a constant factor, is the gravitational potential at synchronous height on the equatorial plane. This involves only the even tesseral harmonics (i.e. those with $\ell - m$ even) since the odd harmonics vanish on the equatorial plane. Almost certainly the principal contribution is from the term involving $J_{2,2}$ which corresponds to an ellipticity of the Earth's equator. If the satellite is sufficiently near to synchronism it can be captured and oscillate within a trough of this potential.

In the main part of this Report these results are extended to orbits at general inclination and which are not necessarily circular. In its motion relative to the Earth such a satellite performs a figure-of-eight, which is

symmetric about the equator if the orbit is circular but is otherwise a distorted three-dimensional curve. The mathematical method adopted is to expand the disturbing function in terms of the usual elliptic elements and to retain only the 'resonant' or long-period terms. Special attention has been given to motion under the $J_{2,2}$ term alone. Finally the analysis is applied to motion under the combination of all tesseral harmonics provided only that the eccentricity is small. It is found that the variation of the mean longitude relative to the Earth is a simple extension of the corresponding result for the circular equatorial orbit, and involves only the introduction of an extra factor $D_{\ell m}(I)$, depending only on the inclination I , for each tesseral harmonic. As before, the motion in mean longitude is equivalent to that of a particle moving in the appropriate one-dimensional potential, and the coefficients $D_{\ell m}(I)$ express how the satellite in its figure-of-eight motion samples the longitudinal force due to each of the tesseral harmonics. Only the even tesseral harmonics are involved since the odd harmonics are antisymmetric about the equator and their effects cancel out.

The analysis presented here should permit the determination of the coefficients of the even tesseral harmonics from observations on synchronous satellites. For at least some of these coefficients, the accuracy should be higher than can currently be obtained from close-orbit satellites. Apart from the converse application in predicting or controlling the motion of future synchronous satellites, such results, possibly taken in conjunction with close-earth results, should improve our knowledge of the shape of the geoid and the Earth's external gravitational field.

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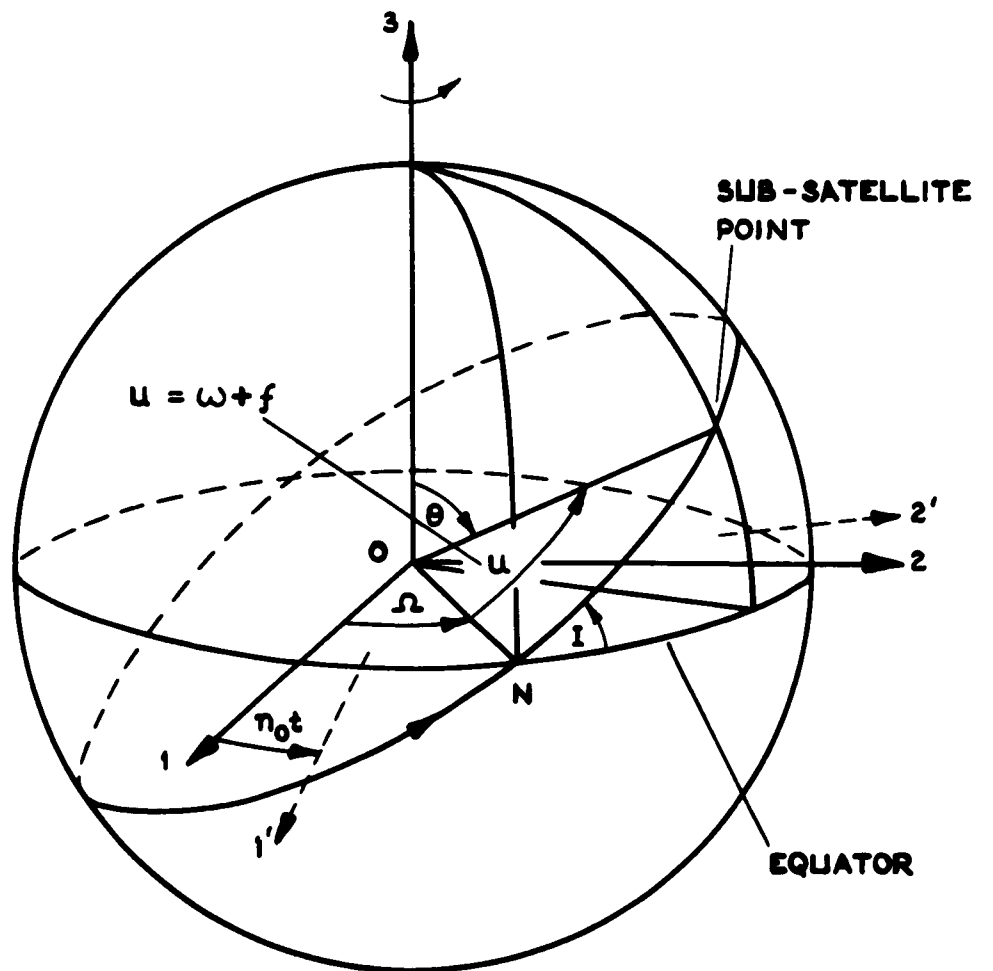


FIG. 1 PROJECTION OF THE SATELLITE ORBIT
ON THE UNIT SPHERE

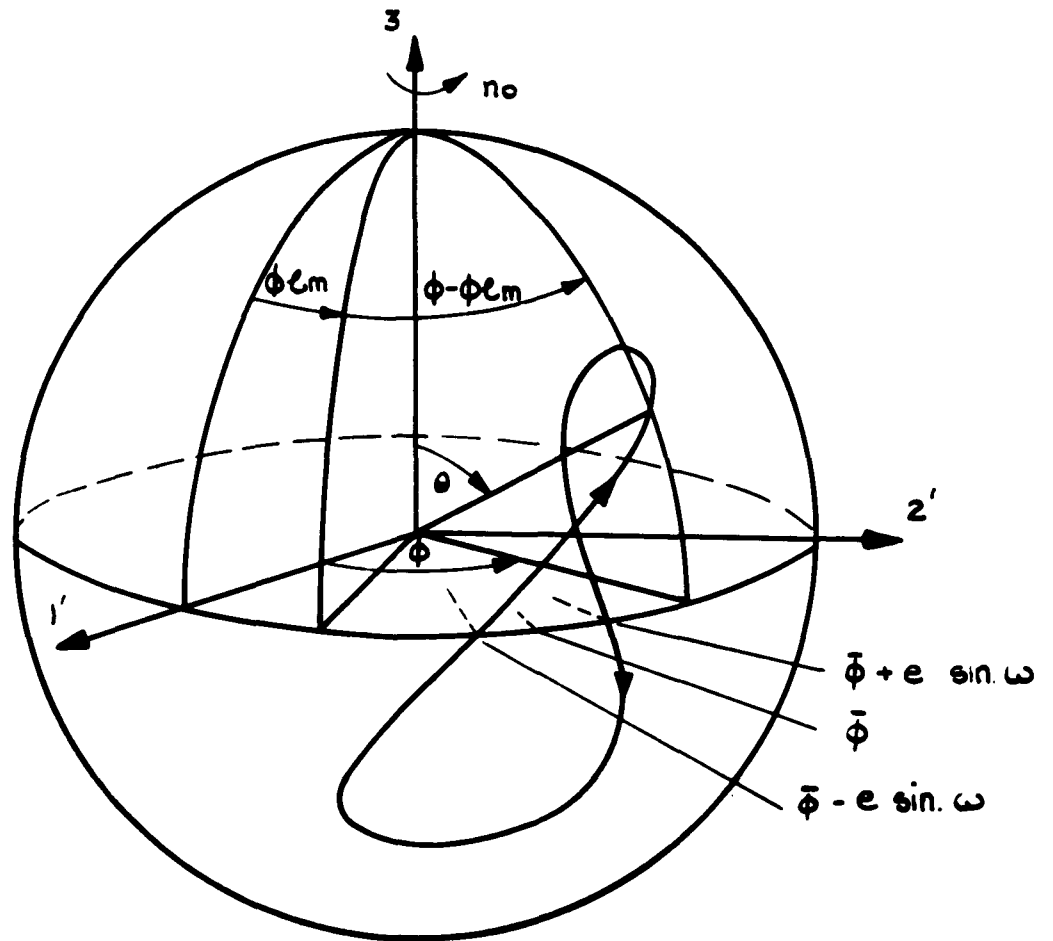


FIG. 2 GROUND TRACK OF SYNCHRONOUS SATELLITE ON THE ROTATING EARTH.

<p>Allan, R.R.</p> <p>ON THE MOTION OF NEARLY SYNCHRONOUS SATELLITES</p> <p>Royal Aircraft Establishment Technical Report 64078</p> <p>December 1964</p> <p>629.195 : 521.26 : 521.6</p> <p>The longitudinal forces due to the longitude-dependent part of the Earth's gravitational potential can have considerable effects on nearly synchronous satellites. As a preliminary the special case of the circular equatorial orbit is considered. An approximate solution is given for the motion under the combination of all tesseral harmonics, and the relation to the Jacobi integral is discussed. For general orbits the disturbing function for the combination of all tesseral harmonics is developed in terms of the usual elliptic elements and the resonant terms identified and isolated. With a suitable definition of the mean longitude, it is shown that the motion in longitude relative to the Earth is equivalent to that of a particle moving in a one-dimensional potential, provided only that the eccentricity is small.</p>	<p>Allan, R.R.</p> <p>ON THE MOTION OF NEARLY SYNCHRONOUS SATELLITES</p> <p>Royal Aircraft Establishment Technical Report 64078</p> <p>December 1964</p> <p>629.195 : 521.26 : 521.6</p> <p>The longitudinal forces due to the longitude-dependent part of the Earth's gravitational potential can have considerable effects on nearly synchronous satellites. As a preliminary the special case of the circular equatorial orbit is considered. An approximate solution is given for the motion under the combination of all tesseral harmonics, and the relation to the Jacobi integral is discussed. For general orbits the disturbing function for the combination of all tesseral harmonics is developed in terms of the usual elliptic elements and the resonant terms identified and isolated. With a suitable definition of the mean longitude, it is shown that the motion in longitude relative to the Earth is equivalent to that of a particle moving in a one-dimensional potential, provided only that the eccentricity is small.</p>
<p>Allan, R.R.</p> <p>ON THE MOTION OF NEARLY SYNCHRONOUS SATELLITES</p> <p>Royal Aircraft Establishment Technical Report 64078</p> <p>December 1964</p> <p>629.195 : 521.26 : 521.6</p> <p>The longitudinal forces due to the longitude-dependent part of the Earth's gravitational potential can have considerable effects on nearly synchronous satellites. As a preliminary the special case of the circular equatorial orbit is considered. An approximate solution is given for the motion under the combination of all tesseral harmonics, and the relation to the Jacobi integral is discussed. For general orbits the disturbing function for the combination of all tesseral harmonics is developed in terms of the usual elliptic elements and the resonant terms identified and isolated. With a suitable definition of the mean longitude, it is shown that the motion in longitude relative to the Earth is equivalent to that of a particle moving in a one-dimensional potential, provided only that the eccentricity is small.</p>	<p>Allan, R.R.</p> <p>ON THE MOTION OF NEARLY SYNCHRONOUS SATELLITES</p> <p>Royal Aircraft Establishment Technical Report 64078</p> <p>December 1964</p> <p>629.195 : 521.26 : 521.6</p> <p>The longitudinal forces due to the longitude-dependent part of the Earth's gravitational potential can have considerable effects on nearly synchronous satellites. As a preliminary the special case of the circular equatorial orbit is considered. An approximate solution is given for the motion under the combination of all tesseral harmonics, and the relation to the Jacobi integral is discussed. For general orbits the disturbing function for the combination of all tesseral harmonics is developed in terms of the usual elliptic elements and the resonant terms identified and isolated. With a suitable definition of the mean longitude, it is shown that the motion in longitude relative to the Earth is equivalent to that of a particle moving in a one-dimensional potential, provided only that the eccentricity is small.</p>